THE FUNDAMENTAL GROUP OF A $\mathbb{CP}^2$ COMPLEMENT OF A BRANCH CURVE AS AN EXTENSION OF A SOLVABLE GROUP BY A SYMMETRIC GROUP

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Abstract. The main result in this paper is as follows:

Theorem. *Let S be the branch curve in $\mathbb{CP}^2$ of a generic projection of a Veronese surface. Then $\pi_1(\mathbb{CP}^2 - S)$ is an extension of a solvable group by a symmetric group.*

A group with the property mentioned in the theorem is "almost solvable" in the sense that it contains a solvable normal subgroup of finite index. We pose the following question.

Question. *For which families of simply connected algebraic surfaces of general type is the fundamental group of the complement of the branch curve of a generic projection to $\mathbb{CP}^2$ an extension of a solvable group by a symmetric group?*

Introduction. Our study of fundamental groups of complements of branch curves is part of our plan to use fundamental groups in order to distinguish among different components of moduli spaces of surfaces of general type.

There are not many known computations of fundamental groups of complements of branch curves. The topic started with Zariski who proved in the 30's that if X is a cubic surface in $\mathbb{CP}^3$ and S is the branch curve of a generic projection of X then $\pi_1(\mathbb{CP}^2 - S) \simeq Z_2 \star Z_3$ (see [Z]). In the late 70's Moishezon proved that if X is a deg n surface in $\mathbb{CP}^3$ then $\pi_1(\mathbb{CP}^2 - S) \simeq B_n/$ Center, where B_n is the braid group of order n (see [Mo]. In fact, Moishezon's result for $n = 3$ is the same as Zariski's

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result since $B_3/\operatorname{Center} \simeq Z_2 \star Z_3$. The next example was Veronese of order 2 (see [MoTe3]). In all the above examples $\pi_1(\mathbb{CP}^2 - S)$ contains a free noncommutative subgroup with 2 generators, so it is "big".

Unlike in the early results, in this paper we present $\pi_1(\mathbb{CP}^2 - S)$ not "big". We study here the fundamental group of the complement in $\mathbb{CP}^2$ of the branch curve of a generic projection of a Veronese surface.

Our main result is as follows:

Theorem. *Let S be the branch curve in $\mathbb{CP}^2$ of a generic projection of a Veronese surface. Then $\pi_1(\mathbb{CP}^2 - S)$ is an extension of a solvable group by a symmetric group.*

We believe that the statement of the theorem is valid for many classes of surfaces of general type. A group with the property mentioned in the theorem is "almost solvable" in the sense that it contains a solvable normal subgroup of finite index. We pose the following question.

Question. *For which families of simply connected algebraic surfaces of general type is the fundamental group of the complement of the branch curve of a generic projection to $\mathbb{CP}^2$ an extension of a solvable group by a symmetric group?*

In [MoTe10], Proposition 2.4, we proved an almost solvability theorem for the complement of S in $\mathbb{C}^2$ where $\mathbb{C}^2$ is a generic affine piece of $\mathbb{CP}^2$. In this paper we move from $\mathbb{C}^2$ to $\mathbb{CP}^2$. This situation involves new techniques (§3 and §5), the Van Kampen Theorem for projective curves, quoted in §1, and different results on the structure of $\pi_1(\mathbb{C}^2 - S)$ from [MoTe9] and [MoTe10] quoted in §4. To formulate the results in §3 and §4 we need some information on the braid group B_n and its quotient $\tilde{B}_n$ which we give in §2. The main theorem is proven in §5.

The theorem can be generalized for any Veronese embedding. In this paper we choose to prove it for an embedding of order 3 to simplify the presentation. The result for any Veronese appears in Section 6.

The braid group B_n plays an important role in describing fundamental groups of complements of curves. There is a quotient of B_n, namely $\tilde{B}_n$, which acts on our group $\pi_1(\mathbb{CP}^2 - S)$. We believe that $\tilde{B}_n$ acts on fundamental groups of complements of branch curves for many classes of surfaces of general type, and we can characterize such fundamental groups through the classification of $\tilde{B}_n$-groups.

Lately, there is a growing interest in fundamental groups in general, in classical algebraic geometry and in Kähler geometry cf., for example, [L], [Si], [To]. For fundamental groups of complements of curves see also [CT] and [DOZ].

§1. The Van Kampen Theorem.

As we stated in the introduction, our starting point for proving the main theorem is the Van Kampen Theorem for projective complements of curves. The Van Kampen Theorem from the 1930's deals with fundamental groups of affine and projective complements of curves. Since our main result is a statement on the fundamental group of the complement in $\mathbb{CP}^2$, we shall only quote in this section the Van Kampen Theorem for the projective complement. We shall start with a few definitions, that we need in order to formulate the theorem.

1.1 Definition. $\underline{\ell(\gamma)}$.

Let D be a disk. Let $p \in \mathrm{Int}(D)$. Let $u \in \partial D$. Let γ be a simple path connecting u with p. We assign to γ a loop as follows: Let c be a small (oriented) circle around p. Let γ' be the part of γ outside of c. We define $\ell(\gamma) = \gamma' \cup c \cup \gamma'^{-1}$. We also use the same notation $\ell(\gamma)$ for the element of $\pi_1(D - K, u)$ corresponding to $\ell(\gamma)$. If $p \in K$, $K \subset D$, K finite, and γ does not meet any other point of K, then $\ell(\gamma)$ can be chosen to be in $\pi_1(D - K, u)$.

1.2 Definition. $\underline{g\text{-base (good geometric base)}}$

Let D be a disk, $K \subseteq D$, $K = \{a_1, \ldots, a_m\}$. Let $u \in \partial(D) - K$. Let $\{\gamma_i\}_{i=1}^m$ be a bush in (D, K, u), i.e., γ_i is a simple path connecting u with a_i, $\forall i, j$ $\gamma_i \cap \gamma_j = u$, $\forall i$ $\gamma_i \cap K =$ one point, and $\{\gamma_i\}$ are ordered counterclockwise around u. Let $\Gamma_i =$

$\ell(\gamma_i) \in \pi_1(D-K,u)$ be the loop around a_i determined by γ_i. $\{\Gamma_i\}_{i=1}^m$ is a g-base of $\pi_1(D-K,u)$.

1.3 Remark. A g-base is a free base of $\pi_1(D-K,*)$ which is essential in the formulation of the Van Kampen Theorem.

1.4. Consider the following situation: Let S be a curve in $\mathbb{CP}^2$ of $\deg m$, s.t. S is transversal to the line in infinity. Let $\pi : \mathbb{C}^2 \to \mathbb{C}$ be a generic projection. Let $N = \{x \in \mathbb{C} \mid \#\pi^{-1}(x) \cap S \underset{\neq}{<} m\}$. Let $u \in \mathbb{C} - N$, s.t. u is real and $|x| < u$ $\forall x \in N$. Let $\mathbb{C}_u = \pi^{-1}(u)$. Let $\{\Gamma_i\}_{i=1}^m$ be a g-base of $\pi_1(\mathbb{C}_u - S \cap \mathbb{C}_u, *)$. By abuse of notation we also use the notation Γ_i for the image of Γ_i in $\pi_1(\mathbb{C}^2 - S, *)$.

1.5 Theorem. (Projective Van Kampen Theorem) *In the situation of* 1.4 *we have*

$$\pi_1(\mathbb{CP}^2 - S, *) \simeq \frac{\pi(\mathbb{C}^2 - S, *)}{\left\langle \prod\limits_{i=1}^m \Gamma_i \right\rangle}.$$

where $\left\langle \prod\limits_{i=1}^m \Gamma_i \right\rangle$ *is the subgroup normally generated by* $\prod\limits_{i=1}^m \Gamma_i$.

Proof. [VK]

§2. Introducing $\tilde{B}_n$, a quotient of B_n.

In this section we bring the definition of the braid group and we distinguish certain elements, called half-twists. Using half-twists we present Artin's Structure Theorem for the braid group and the natural homomorphism to the symmetric group. We also define transversal half-twists and the quotient of B_n called $\tilde{B}_n$.

2.1 Definition. Braid group $B_n = B_n[D,K]$

Let D be a closed disc in $\mathbb{R}^2$, $K \subset D$, K finite. Let B be the group of all diffeomorphisms β of D such that $\beta(K) = K$, $\beta|_{\partial D} = \mathrm{Id}_{\partial D}$. For $\beta_1, \beta_2 \in B$, we say that β_1 is equivalent to β_2 if β_1 and β_2 induce the same automorphism of $\pi_1(D-K,u)$. The quotient of B by this equivalence relation is called the braid group $B_n[D,K]$ $(n = \#K)$. The elements of $B_n[D,K]$ are called braids.

2.2 Definition. $H(\sigma)$, half-twist defined by σ

Let D, K be as above. Let $a, b \in K$, $K_{a,b} = K - a - b$ and σ be a simple path in $D - \partial D$ connecting a with b s.t. $\sigma \cap K = \{a, b\}$. Choose a small regular neighborhood U of σ and an orientation preserving diffeomorphism $f : \mathbb{R}^2 \to \mathbb{C}^1$ ($\mathbb{C}^1$ is taken with the usual "complex" orientation) such that $f(\sigma) = [-1, 1]$, $f(U) = \{z \in \mathbb{C}^1 \mid |z| < 2\}$. Let $\alpha(r), r \geq 0$, be a real smooth monotone function such that $\alpha(r) = 1$ for $r \in [0, \frac{3}{2}]$ and $\alpha(r) = 0$ for $r \geq 2$.

Define a diffeomorphism $h : \mathbb{C}^1 \longrightarrow \mathbb{C}^1$ as follows. For $z \in \mathbb{C}^1$, $z = re^{i\varphi}$, let $h(z) = re^{i(\varphi + \alpha(r))}$. It is clear that on $\{z \in \mathbb{C}^1 \mid |z| \leq \frac{3}{2}\}$, $h(z)$ is the positive rotation by 180° and that $h(z) =$ Identity on $\{z \in \mathbb{C}^1 \mid |z| \geq 2\}$, in particular, on $\mathbb{C}^1 - f(U)$. Considering $(f \circ h \circ f^{-1})|_D$ (we always take composition from left to right), we get a diffeomorphism of D which interchanges a and b and is the identity on $D - U$. Thus it defines an element of $B_n[D, K]$, called the half-twist defined by σ and denoted $H(\sigma)$.

Using half-twists we build a set of generators for B_n.

2.3 Definition. Frame of $B_n[D, K]$

Let D be a disc in $\mathbb{R}^2$. Let $K = \{a_1, \dots, a_n\}$, $K \subset D$. Let $\sigma_1, \dots, \sigma_{n-1}$ be a system of simple paths in $D - \partial D$ such that each σ_i connects a_i with a_{i+1} and for

$$i, j \in \{1, \dots, n-1\}\ ,\ i < j \quad , \quad \sigma_i \cap \sigma_j = \begin{cases} \emptyset & \text{if } |i - j| \geq 2 \\ a_{i+1} & \text{if } j = i + 1\,. \end{cases}$$

Let $H_i = H(\sigma_i)$. We call the ordered system of half-twists $(H_1, \dots, H_{n-1})$ a frame of $B_n[D, K]$ defined by $(\sigma_1, \dots, \sigma_{n-1})$, or a frame of $B_n[D, K]$ for short.

2.4 Notation.

$[A, B] = ABA^{-1}B^{-1}$.

$\langle A, B \rangle = ABAB^{-1}A^{-1}B^{-1}$.

$(A)_B = B^{-1}AB$.

2.5 Theorem. (E. Artin's braid group presentation) *Let $\{H_i\}$ be a frame of B_n. Then B_n is generated by the half-twists H_i and all the relations between $H_1, \dots, H_{n-1}$ follow from*

$$[H_i, H_j] = 1 \qquad \text{if} \quad |i-j| > 1,$$
$$\langle H_i, H_j \rangle = 1 \qquad \text{if} \quad |i-j| = 1,$$
$$1 \le i, j \le n-1.$$

Proof. [A] (or [MoTe4], Chapter 5).

2.6 Theorem. *Let $\{H_i\}$ be a frame of B_n. Then*

(i) *for $n \ge 2$,* Center B_n *is isomorphic to Z with a generator $\Delta_n^2 = (H_1 \cdot \ldots \cdot H_{n-1})^n$.*

(ii) *$B_2 \simeq \mathbb{Z}$ with a generator H_1.*

Proof. [MoTe4], Corollary V.2.3.

2.7 Proposition. *There is a natural defined homomorphism $B_n \to S_n$ (symmetric group on n elements) defined by $H_i \to (i \; i+1)$.*

Proof. Since the transpositions $\alpha_i = (i \; i+1)$ satisfy the relations from Artin's theorem (2.5), the above homomorphism is well defined.

2.8 Definition. $\underline{P_n}$.

The kernel of the above homomorphism is denoted by P_n.

2.9 Remark. The transpositions α_i satisfy a relation that H_i do not satisfy, which is $\alpha_i^2 = 1$. In fact it is true for any transposition. Under the above homomorphism the image of any half-twist is a transposition and thus any square of a half-twist belongs to $\ker(B_n \to S_n)$ which is P_n.

2.10 Definition. <u>Transversal half-twists, adjacent half-twist, disjoint half-twist</u>.

Let σ_1 and σ_2 be 2 paths in D with endpoints in K which do not intersect K otherwise (like in 2.2). The half-twists $H(\sigma_1)$ and $H(\sigma_2)$ will be called *transversal*

if σ_1 and σ_2 intersect transversally in one point which is not an end point of either of the σ_i's.

The half-twists $H(\sigma_1)$ and $H(\sigma_2)$ will be called *adjacent* if σ_1 and σ_2 have one endpoint in common.

The half-twists $H(\sigma_1)$ and $H(\sigma_2)$ will be called *disjoint* if σ_1 and σ_2 do not intersect.

2.12 Claim. *Disjoint half-twists commute and adjacent half-twists satisfy the triple relation $ABA = BAB$.*

Proof. By Proposition 2.7 and the fact that every 2 half-twists are conjugated to each other.

2.12 Definition. $\underline{\tilde{B}_n}$.

Let Q_n be the subgroup of B_n normally generated by $[X, Y]$ for X, Y transversal half-twists. $\tilde{B}_n$ is the quotient of B_n modulo Q_n. For $X \in B_n$ we denote by $\tilde{X}$ the image of X in $\tilde{B}_n$. $\{\tilde{H}_i\}$ is a frame of $\tilde{B}_n$ if $\{H_i\}$ is a frame of B_n.

Later we shall need some basic relations satisfied in $\tilde{B}_n$ (and not in B_n). We formulate this in the following claim.

2.13 Claim. *Let $\tilde{P}_n$ be the image of P_n (from 2.7) in $\tilde{B}_n$. Then $\tilde{P}'_n = \{1, c\}$ where $c^2 = 1$, $c \in \text{Center}\, \tilde{B}_n$. In particular, if $\tilde{X}$ and $\tilde{Y}$ are 2 adjacent half-twists $[\tilde{X}^{\pm 2}, \tilde{Y}^{\pm 2}] = c$.*

Proof. [MoTe9], Proposition II.5.2.

§3. General results on fundamental groups of complements of curves.

In this section we prove two general results based on the situation described in 1.4. The first one concerns the action of the braid group on the fundamental group $\pi_1(\mathbb{C}^2 - S)$ and the second one is a corollary on the structure of $\pi_1(\mathbb{C}^2 - S)$.

3.1 Proposition. *Consider the situation of 1.4. Let Δ_m^2 be the generator of the center of $B_m[\mathbb{C}_u, \mathbb{C}_u \cap S]$. Then when considered as elements of $\pi_1(\mathbb{C}^2 - S)$,*

$\Delta_m^2(\Gamma_k) = \Gamma_k \quad \forall \Gamma_k$.

Proof. Let φ_u be the naturally defined homomorphism from $\pi_1(\mathbb{C} - N, u) \to B_m[\mathbb{C}_u, \mathbb{C}_u \cap S]$. This homomorphism is called the braid monodromy and it factors through the classical monodromy from π_1 to S_m, $\underbrace{\pi_1 \overset{\varphi_u}{\to} B_m \to S_m}_{\psi}$. Since B_m acts on $\pi_1(\mathbb{C}_u - S, *)$, so does $\varphi_u(\pi_1)$. Moreover, $\varphi_u(\pi_1)$ acts on the elements of $\pi_1(C_u - S, *)$ when considered as elements of $\pi_1(\mathbb{C}^2 - S, *)$. By the affine Van Kampen theorem (see [RoTe]), for $\gamma \in \pi_1(\mathbb{C} - N, u)$, $\varphi_u(\gamma)(\Gamma_k) = \Gamma_k$ when considered as elements of $\pi_1(\mathbb{C}^2 - S)$. By [MoTe4], Lemma VI.2.1, Δ_m^2 is a product of elements of the form $\varphi_u(\gamma)$ for $\gamma \in \pi_1(\mathbb{C} - N, u)$ and thus $\Delta_m^2(\Gamma_k) = \Gamma_k \ \forall \Gamma_k$.

3.2 Proposition. *Consider the situation of* 1.4. *Let* $\Gamma = \prod\limits_{i=1}^{m} \Gamma_i$. *Then when* Γ *is considered as an element of* $\pi_1(\mathbb{C}^2 - S, *)$, *it is a central element and* $\langle \Gamma \rangle$ *is an infinite cyclic group.*

Proof. We first consider $\Gamma = \prod\limits_{i=1}^{m} \Gamma_i$ as an element of $\pi_1(\mathbb{C}_u - S, *)$. Clearly, $\prod \Gamma_i$ is homotopic to a loop ∂D around all the points of $\mathbb{C}_u \cap S$. By Proposition V.2.1 of [MoTe5] in $\pi_1(\mathbb{C}_u - S, *)$ the conjugation of Γ_k by ∂D is equal to the action of Δ_m^2 (which is defined in 2.6) on Γ_k. But in $\pi_1(\mathbb{C}^2 - S, *)$, Δ_m^2 acts trivially on Γ_k, (by 3.1) so conjugation of Γ_k by ∂D in $\pi_1(\mathbb{C}^2 - S)$ is stable and thus $\Gamma = \partial D$ is in the center of $\pi_1(\mathbb{C}^2 - S, u)$. Moreover, since $\langle \Delta_m^2 \rangle$ is an infinite cyclic group (see ([MoTe4], V.2.1)), so is $\langle \Gamma \rangle$.

§4. Results on $\pi_1(\mathbb{C}^2 - S, *)$ for a Veronese branch curve.

In this section we restrict ourselves to a curve which is the branch curve of a Veronese generic projection. We will quote results concerning its complement in $\mathbb{C}^2$ (cf. [MoTe9] and [MoTe10]) which will be used later in the proof of the main result, concerning its complement in $\mathbb{CP}^2$.

The fundamental group of the complement in $\mathbb{C}^2$ turned out to be a quotient of a semidirect product of $\tilde{B}_{n^2}$ (for a Veronese embedding of $\deg n$) and $G_0(n^2)$ which is a $\mathbb{Z}_2$ extension of a free group on $n^2 - 1$ elements (see [MoTe9]).

From now on we will restrict ourselves to a Veronese embedding of deg 3. Let S be the branch curve of a generic projection to $\mathbb{CP}^2$ of a Veronese surface of deg 3. The degree of the projection is 9, and the degree of the branch curve is 18. Let $\mathbb{C}^2$ be a big affine piece of $\mathbb{CP}^2$ s.t. S is transversal to the line in infinity.

Consider $\tilde{B}_9$ as defined in §2. Instead of working with a frame of $\tilde{B}_9$ we will work with $\{\tilde{T}_i\}$, a set of generators for $\tilde{B}_9$ as follows:

4.1 Definition. Let $\{T_i\}_{i=1\ i\neq 4}^{9}$ be s.t. $\tilde{T}_i$ is a half-twist w.r.t. t_i where t_i are arranged as in Figure 4.1

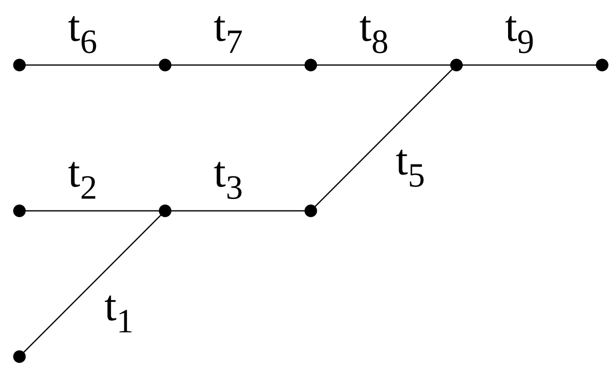

FIGURE 4.1

4.2 Remark. The choice of the base originates from a configuration of planes in the degeneration of V_3 to a union of planes. We constructed this degeneration in [MoTe7], but we do not use it directly in this paper. It was used in [MoTe9] to prove the results which are quoted here.

4.3. It is easy to see that

T_i and T_j are adjacent for (i, j) as follows:

$i, j \in \{1, 2, 3\}$

$i = 5 \qquad j = 3, 8, 9$

$i = 6, 7, 8 \qquad j = i + 1.$

T_i and T_j are disjoint for (i, j) as follows:

$i \in \{1, 2, 3\} \quad j \in \{6, 7, 8, 9\}$

$i = 5 \qquad j = 1, 2, 6, 7$

$i = 6 \qquad j = 8, 9$

$i = 7 \qquad j = 9.$

4.4 Claim. *The set $\{T_i\}$ satisfies the following relations:*

$$\langle T_i, T_j\rangle = 1 \qquad \textit{if } T_i \textit{ and } T_j \textit{ are adjacent}$$

$$[T_i, T_j] = 1 \qquad \textit{if } T_i \textit{ and } T_j \textit{ are disjoint}$$

$$[T_1, T_2^{-1}T_3T_2] = 1$$

$$[T_5, T_8^{-1}T_9T_8] = 1$$

Proof. Since the sequence of half-twist $\left\{T_1, T_2, T_2^{-1}T_3T_2, T_5, T_9, T_8^{-1}\ T_9\ T_8, T_7, T_6\right\}$ is represented by a consecutive sequence of paths (see Fig. 4.2), it is a frame. By E. Artin's Theorem, they satisfy the relations that a frame satisfies (Theorem 2.5). When writing down the triple relations for the above frame, we get

$$\langle T_1, T_2\rangle = 1$$

$$\langle T_2, T_2^{-1}T_3T_2\rangle = 1$$

$$\langle T_2^{-1}T_3T_2, T_5\rangle = 1$$

$$\langle T_5, T_9\rangle = 1$$

$$\langle T_9, T_8^{-1}T_9T_8\rangle = 1$$

$$\langle T_8^{-1}T_9T_8, T_7\rangle = 1$$

$$\langle T_7, T_6\rangle = 1$$

When writing down the commutative relations, we get:

$$[T_i, T_j] = 1 \text{ for } T_i \text{ or } T_j \text{ disjoint plus}$$

$$[T_1, T_2^{-1}T_3T_2] = 1$$

$$[T_5, T_8^{-1}T_9T_8] = 1.$$

We just need to show $\langle T_8, T_9\rangle = 1$, $\langle T_2, T_3\rangle = 1$. Since T_8 and T_9 are adjacent, by Claim 2.11, $T_8^{-1}T_9T_8 = T_9T_8T_9^{-1}$. Now from $\langle T_9, T_8^{-1}T_9T_8\rangle = 1$, we get

$$\begin{aligned} 1 &= T_9T_8^{-1}T_9T_8T_9T_9T_8^{-1}T_9^{-1}T_9T_9^{-1}T_8^{-1}T_9^{-1} = \\ &= T_9T_9T_8T_9^{-1}T_9T_9T_8^{-1}T_9^{-1}T_8^{-1}T_9^{-1} \\ &= T_9T_9T_8T_9T_8^{-1}T_9^{-1}T_8^{-1}T_9^{-1} \end{aligned}$$

Thus also $T_9T_8T_9T_8^{-1}T_9^{-1}T_8^{-1} = 1$. Thus $\langle T_9, T_8\rangle = 1$. Similarly, $\langle T_2, T_3\rangle = 1$, and we get the claim.

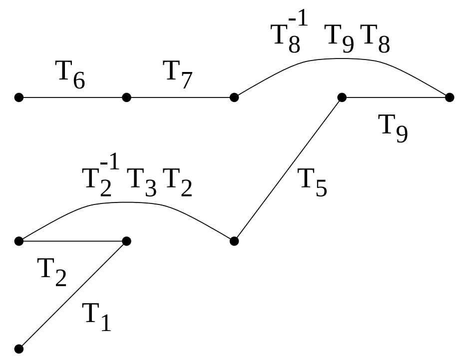

FIGURE 4.2

4.5 Definition. Polarization, orderly adjacent, non orderly adjacent.

We choose an orientation on each T_i with compatibility with its "bigger" neighbor. We call it a polarization. See Figure 4.3.

Most of the adjacent T_i's are orderly adjacent (compatible polarization) apart from $\{T_1, T_2\}$ and $\{T_5, T_8\}$ which are non orderly adjacent.

4.6 Definition. $G_0(9)$.

$G_0(9)$ is a $\mathbb{Z}_2$ extension of a free group on 8 elements. We take the following model for $G_0(9)$:

Let $G_0(9)$ be generated by $\{g_i\}_{i=1\ i\neq 4}^{9}$ s.t.

$$[g_i, g_j] = \begin{cases} 1 & T_i, T_j \text{ are disjoint} \\ \tau & \text{otherwise} \end{cases}$$

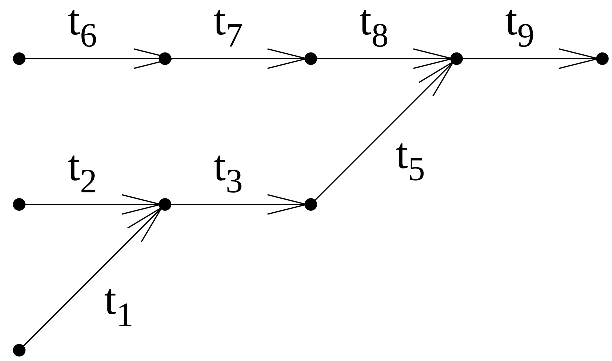

FIGURE 4.3

where $\tau^2 = 1$, $\tau \in \mathrm{Center}\, G_0(9)$.

We take the following action of $\tilde{B}_9$ on $G_0(9)$

$$(g_i)_{\tilde{T}_k} = \begin{cases} g_i^{-1}\tau & k = i \\ g_i & T_i, T_k \text{ are disjoint} \\ g_i g_k^{-1} & T_i, T_k \text{ are not orderly adjacent} \\ g_k g_i & \text{otherwise} \end{cases}$$

4.7 Definition. $\underline{G_9,\ c}$.

Consider the semidirect product $\tilde{B}_9 \ltimes G_0(9)$ w.r.t. the chosen action.

Let $c = [\tilde{T}_1^2, \tilde{T}_2^2]$.

Let $\xi_1 = (\tilde{T}_2\tilde{T}_1\tilde{T}_2^{-1})^2\tilde{T}_2^{-2}$.

Let $N_9 \triangleleft \tilde{B}_9 \ltimes G_0(9)$ be normally generated by $c\tau^{-1}$ and $(g_1\xi_1^{-1})^3$.

Let $G_9 = \dfrac{\tilde{B}_9 \ltimes G_0(9)}{N_9}$.

4.8 Definition. $\underline{\hat{\psi}_9}$.

Let $\tilde{\psi}_9$ be the homomorphism $\tilde{B}_9 \to S_9$ induced from the standard homomorphism $B_9 \to S_9$ (see 2.7). $\tilde{\psi}_9$ exists since $[X, Y] \to 1$ under the standard homomorphism. Let $\hat{\psi}_9 : G_9 \to S_9$ be defined by the first coordinate $\hat{\psi}_9(\alpha, \beta) = \tilde{\psi}_9(\alpha)$.

4.9 Definition. $\underline{\psi}$.

The projection $V_3 \to \mathbb{C}^2$, of degree 9, induces a standard monodromy homomorphism $\pi_1(\mathbb{C}^2 - S, *) \to S_9$ which we denote by ψ.

4.10 Proposition. *$\pi_1(\mathbb{C}^2 - S, *) \simeq G_9$ s.t. ψ is compatible with $\hat{\psi}_9$.*

Proof. [MoTe9], VI.1.

4.11 Definition. $\underline{H_9, H_{9,0}, H'_9, H'_{9,0}}$.

Let $Ab : B_9 \to \mathbb{Z}$ be the abelianization of B_9 and B_9 over its commutator subgroup.

Let $\widetilde{A_b} : \tilde{B}_9 \to \mathbb{Z}$ be a homomorphism induced from Ab (which exists since $Ab([X, Y]) = 1$).

Let $\widehat{Ab} : G_9 \to \mathbb{Z}$ be defined by the first coordinate $\widehat{Ab}(\alpha, \beta) = \widetilde{Ab}(\alpha)$.

Let $H_9 = \ker \hat{\psi}_9$.

Let $H_{9,0} = \ker \hat{\psi}_9 \cap \ker \widehat{Ab}$.

Let $H'_9, H'_{9,0}$ be the commutant subgroup of H_9 and $H_{9,0}$ respectively.

4.12 Proposition. *There exists a series* $1 \triangleleft H'_{9,0} \triangleleft H_{9,0} \triangleleft H_9 \triangleleft G_9$, *where* $G_9/H_9 \simeq S_9$, $H_9/H_{9,0} \simeq \mathbb{Z}$, $\quad H_{9,0}/H'_{9,0} \simeq (\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z})^8$, $\quad H'_{9,0} = H'_9 \cong \mathbb{Z}/2\mathbb{Z}$.

Proof. [MoTe10], Proposition 2.4.

Our main result is a result of type 4.12.

To this end we need to get into the proofs of the structure theorems for G_9, which are quoted in 4.10 and 4.12. We need this for the proof of the main result

4.13. $H_{9,0}$ is generated by $\{g_i\}_{i=1\ i\neq 4}^{9}$, $\quad \{\xi_i\}_{i=1\ i\neq 4}^{9}$, $\quad c$ where

$$[g_i, g_j] = \begin{cases} 1 & T_i, T_j \text{ are disjoint} \\ c & \text{otherwise.} \end{cases}$$

$$[\xi_i, \xi_j] = \begin{cases} 1 & T_i, T_j \text{ are disjoint} \\ c & \text{otherwise.} \end{cases}$$

$$[\xi_i, g_j] = \begin{cases} 1 & T_i, T_j \text{ are disjoint} \\ c & \text{otherwise.} \end{cases}$$

g_i, ξ_i of infinite order.

$g_i^3 = \xi_i^3$.

$c^2 = 1$.

$c \in \text{Center}(G_9)$.

H_9 is generated by $H_{9,0}$ and $\tilde{T}_1^2$ where $\tilde{T}_1^2$ is of infinite order.

$H_9' = H_{9,0}'$ is generated by c. (c is the image of the generator of $\tilde{P}_9'$ from 2.12).

For simplicity we also denote $\zeta_i = g_i \xi_i^{-1}$, ($\zeta_i^3 = 1$).

From 4.13 we get the following:

4.14. $H_{9,0}' = H_9' \simeq \mathbb{Z}_2 (\subseteq \text{Center}(G_9))$.

$H_{9,0}/H_{9,0}'$ is generated by $\{\xi_i\}_{i=1,i\neq 4}^9$ and $\{\zeta_i\}_{i=1,i\neq 4}^9$, when the only relations are the commutativity relation and $\zeta_i^3 = 1$. Thus $\dfrac{H_{9,0}}{H_{9,0}'} \simeq (\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z})^8$.

$H_9/H_{9,0}$ is generated by $\tilde{T}_1^2$ and thus is isomorphic to $\mathbb{Z}$.

H_9 is the kernel of $G_9 \to S_9$, and thus $G_9/H_9 \simeq S_9$.

§5. The Main Result.

Our main result is the following theorem.

5.0 Theorem. *Let S be the branch curve of a generic projection to $\mathbb{CP}^2$ of a Veronese embedding of* $\deg 3$. *Then $\pi_1(\mathbb{CP}^2 - S, *)$ is an extension of a solvable group by the symmetric group of* 9 *elements. In fact, we have* $1 \triangleleft \overline{H}_{9,0}' \triangleleft \overline{H}_{9,0} \triangleleft \overline{H}_9 \triangleleft \overline{G}_9$ *where* $\overline{G}_9/\overline{H}_9 \simeq S_9$, $\overline{H}_9/\overline{H}_{9,0} \simeq Z_9$, $\overline{H}_{9,0}/\overline{H}_{9,0}' \simeq (\mathbb{Z} \oplus \mathbb{Z}_3)^8$, $\overline{H}_{9,0}' \simeq \mathbb{Z}_2$.

Proof. It is easy to calculate $\deg S$ (see [MoTe3]) and it is 18.

We consider the situation of 1.4 for the branch curve from our Theorem. By [MoTe9], Lemma 2.3, there is a possibility to choose a g-base $\{\Gamma_i, \Gamma_{i'}\}_{i=1}^9$ s.t. $\psi(\Gamma_i) = \psi(\Gamma_{i'}) =$ transposition. (This choice is a consequence of the degeneration of the surface to a union of 9 planes.)

By 1.5, $\pi_1(\mathbb{CP}^2 - S, *) = \dfrac{\pi_1(\mathbb{C}^2 - S, *)}{\left\langle \prod\limits_{i=9}^{1} \Gamma_{i'}\Gamma_i \right\rangle}$. Denote $\hat{\beta} : \pi_1(\mathbb{CP}^2 - S, *) \to G_9$ to be the isomorphism from 4.10 and $\delta = \hat{\beta}\left(\prod\limits_{i=9}^{1} \Gamma_{i'}\Gamma_i\right)$. Clearly, $\pi_1(\mathbb{CP}^2 - S, *) \simeq \dfrac{G_9}{\langle\delta\rangle}$ which we denote by $\overline{G}_9$. To prove the theorem we shall prove that $\overline{G}_9 = \dfrac{G_9}{\langle\delta\rangle}$ is an extension of a solvable group by a symmetric group.

In 4.12 we introduced a sequence $1 \triangleleft H'_{9,0} \triangleleft H_{9,0} \triangleleft H_9 \triangleleft G_9$ and the appropriate quotients. Let $\overline{H}'_{9,0}$, $\overline{H}_{9,0}$, $\overline{H}_9$ be the images of $H'_{9,0}$, $H_{9,0}, H_9$ in $\overline{G}_9$ respectively. To prove the theorem we shall compute $\overline{G}_9/\overline{H}_9$, $\overline{H}_9/\overline{H}_{9,0}$, $\overline{H}_{9,0}/\overline{H}_{9,0}$ and $\overline{H}'_{9,0}$.

We first need to prove some results on G_9 in general and on δ in particular. This is done in Claims 5.1–5.10. From general arguments we already know that $\delta \in \text{Center}(G_9)$ (cf. Proposition 3.2).

5.1 Claim. $\delta \in H_9$, $\hat{\beta}(\Gamma_i \Gamma_{i'}) \in H_9$.

Proof of Claim 5.1. By 4.10, $\hat{\psi}_9 \hat{\beta}(\Gamma_{i'} \Gamma_i) = \psi_(\Gamma_{i'} \Gamma_i)$. Since $\psi(\Gamma_i) = \psi(\Gamma_{i'}) =$ transposition. $\psi(\Gamma_{i'} \Gamma_i) = 1$, and thus $\hat{\beta}(\Gamma_{i'} \Gamma_i) \in \ker \hat{\psi}_9 = H_9$. Since $\delta = \prod \hat{\beta}(\Gamma_{i'} \Gamma_i)$, it is also in H_9. $\square$ for Claim 5.1

The new quotients will be determined by an expression of δ as a product of elements in $H_{9,0}$ and elements which are in H_9 but not in $H_{9,0}$.

5.2 Definition.

By abuse of notation the images in $G_9 = \dfrac{\tilde{B}_9 \ltimes G_0(9)}{N_9}$, of $\tilde{T}_i$ from $\tilde{B}_9$ (see 4.2) are also denoted by $\tilde{T}_i$. We also define:

$\tilde{T}_4 = (\tilde{T}_5)_{\tilde{T}_8^{-1} \tilde{T}_7 \tilde{T}_3^{-1} \tilde{T}_2}$.

$g_4 = (g_5)_{\tilde{T}_8^{-1} \tilde{T}_7 \tilde{T}_3^{-1} \tilde{T}_2}$ for g_5 from 4.13.

$\xi_4 = (\xi_5)_{\tilde{T}_8^{-1} \tilde{T}_7 \tilde{T}_3^{-1} \tilde{T}_2}$ for ξ_5 from 4.13.

To work in G_9 we need some commutativity relations:

5.3 Claim. *In G_9:*

(i) $\tilde{T}_i^2$, $\xi_i, g_i \in H_9$ $i = 1, \dots, 9$.

(ii) $[\tilde{T}_i^2, g_j], [\xi_i g_i] = 1$ *or* c.

(iii) *If X and Y are 2 adjacent half-twists, then* $[\tilde{X}^2, \tilde{Y}^2] = c$.

Proof.

(i) Since T_i is a half-twist $i = 1 \dots 9$, thus $\hat{\psi}_9(\tilde{T}_i)$ is a transposition and $\hat{\psi}_9(\tilde{T}_i^2) = 1$. Thus $\tilde{T}_i^2 \in H_9$. The elements $\{g_i, \xi_i\}_{i=1\ i\neq 4}^{9}$ are in H_9 by 4.13. Since

g_5 is in H_9 and H_9 is a normal subgroup ($= \ker \hat{\psi}_9$), g_4 is also in H_9. The same applies for ξ_4.

(ii) Since $H'_9 = \{1, c\}$, $c^2 = 1$.

(iii) Since it is true in $\tilde{B}_9$ by Claim 2.13. $\square$ for Claim 5.3

In order to compute the corresponding quotients in $\overline{G}_9$, we need to express δ which is in H_9 (see 5.1) in terms of the following generators of H_9: $\{\zeta_i\}_{i=1\ i\neq 4}^{9}$, $\{\xi_i\}_{i=1\ i\neq 4}^{9}, c$ and $\tilde{T}_1^2$ (see 4.13). Recall that $\delta = \prod\limits_{i=9}^{1} \hat{\beta}(\Gamma_{i'}\Gamma_i)$ where $\forall i = 1 \dots 9$ $\hat{\beta}(\Gamma_{i'}\Gamma_i) \in H_9$. Thus, we shall first express $\hat{\beta}(\Gamma_{i'}\Gamma_i)$ for $i = 1 \dots 9$ in terms of $\{\zeta_i\}_{i=1\ i\neq 4}^{9}$, $\{\xi_i\}_{i=1\ i\neq 4}^{9}, c$ and $\tilde{T}_1^2$, and then we multiply these expressions to get an expression for δ in these generators (see 5.9). In 5.10 we replace g_i by $\zeta_i\xi_i$.

5.4 Claim.

$$\hat{\beta}(\Gamma_{1'}\Gamma_1) = g_1\tilde{T}_1^2$$

$$\hat{\beta}(\Gamma_{2'}\Gamma_2) = g_2^{-1}\xi_2\tilde{T}_2^2$$

$$\hat{\beta}(\Gamma_{3'}\Gamma_3) = g_3\xi_3^{-1}\tilde{T}_3^2$$

$$\hat{\beta}(\Gamma_{4'}\Gamma_4) = g_4^{-1}\xi_4\tilde{T}_4^2$$

$$\hat{\beta}(\Gamma_{5'}\Gamma_5) = g_5^{-1}\xi_5\tilde{T}_5^2$$

$$\hat{\beta}(\Gamma_{6'}\Gamma_6) = g_6\tilde{T}_6^2$$

$$\hat{\beta}(\Gamma_{7'}\Gamma_7) = g_7\xi_7^{-1}\tilde{T}_7^2$$

$$\hat{\beta}(\Gamma_{8'}\Gamma_8) = g_8^{-1}\xi_8\tilde{T}_8^2$$

$$\hat{\beta}(\Gamma_{9'}\Gamma_9) = cg_g^{-1}\tilde{T}_9^2$$

Proof. We take a new set of generators for G:

$$E_i = \begin{cases} \Gamma_i & i \neq 2,7 \\ \Gamma_{i'} & i = 2,7 \end{cases} \quad E'_{i'} = \begin{cases} \Gamma_{i'} & i \neq 2,7 \\ \Gamma_{i'}\Gamma_i\Gamma_{i'}^{-1} & i = 2,7 \end{cases}$$

(This choice which was made in [MoTe9] originated from a certain relation in G induced by the affine Van Kampen Theorem.) Clearly $\Gamma_{i'}\Gamma_i = E_{i'}E_i$. Let $A_i =$

$E_{i'}E_i^{-1}$. Clearly, $\Gamma_{i'}\Gamma_i = E_{i'}E_i = A_iE_i^2$. By the construction of $\hat{\beta}$ (see [MoTe9], Ch.V), $\hat{\beta}(E_i^2) = \tilde{T}_i^2$ and $\hat{\beta}(A_i)$ is as follows:

$$\beta(A_1) = g_1$$

$$\beta(A_2) = g_2^{-1}\xi_2$$

$$\beta(A_3) = g_3\xi_3^{-1}$$

$$\beta(A_4) = g_4^{-1}\xi_4$$

$$\beta(A_5) = g_5^{-1}\xi_5$$

$$\beta(A_6) = g_6$$

$$\beta(A_7) = g_7\xi_7^{-1}$$

$$\beta(A_8) = g_8^{-1}\xi_8$$

$$\beta(A_9) = cg_9^{-1}$$

□ for Claim 5.6

In the next step we express $\tilde{T}_i^2$ in terms of $\{\xi_i\}_{i=1}^9$ and $\tilde{T}_1^2$. The main point in the proof of the next claim is that for 3 half-twists which form a triangle where one of the edges is T_i, the product $\tilde{X}^2\tilde{Y}^{-2}$ of the other 2 half-twists can be expressed in terms of ξ_i, ξ_i^{-1} and c. The exact statement is as follows:

5.5 Claim. *Let X, Y be 2 half-twists, $X = H(x)$, $Y = H(y)$, $T_i = H(t_i)$ s.t. x, y, t_i make a triangle. Assume that x and y meet in ν, and a counterclockwise rotation around ν inside the triangle meets x before it meets y.*

(i) *If the polarization of T_i goes from x to y, then $\xi_i = \tilde{X}^2\tilde{Y}^{-2}$.*

(ii) *If the polarization of T_i goes from y to x, then $\xi_i = \tilde{X}^{-2}\tilde{Y}^2$.*

Proof. Claim IV.4.1 of [MoTe9].

Using this claim we prove the following

5.6 Lemma.

(i) $\tilde{T}_2^2\tilde{T}_1^{-2} = c\xi_1^{-1}\xi_2$

(ii) $\tilde{T}_3^2\tilde{T}_1^{-2} = \xi_1^{-1}\xi_3^{-1}$

(iii) $\tilde{T}_5^2\tilde{T}_1^{-2} = \xi_1^{-1}\xi_3^{-2}\xi_5^{-1}$

(iv) $\tilde{T}_9^2\tilde{T}_1^{-2} = c\xi_1^{-1}\xi_3^{-2}\xi_5^{-2}\xi_9^{-1}$

(v) $\tilde{T}_8^2\tilde{T}_1^{-2} = \xi_1^{-1}\xi_3^{-2}\xi_5^{-2}\xi_8$

(vi) $\tilde{T}_7^2\tilde{T}_1^{-2} = c\xi_1^{-1}\xi_3^{-2}\xi_5^{-2}\xi_7\xi_8^2$

(vii) $\tilde{T}_6^2\tilde{T}_1^{-2} = \xi_1^{-1}\xi_3^{-2}\xi_5^{-2}\xi_6\xi_7^2\xi_8^2$

(viii) $\tilde{T}_4^2\tilde{T}_1^{-2} = \xi_1^{-1}\xi_3^{-2}\xi_4\xi_5^{-2}\xi_7^2\xi_8^2$

Proof. The proof is based on Claim 5.5.

Moreover, we interchange between the ξ_i's using the commutator from 4.13

$$[\xi_i^{\pm 1}, \xi_j^{\pm 1}] = \begin{cases} 1 & T_i \text{ and } T_j \text{ are disjoint} \\ c & \text{otherwise} \end{cases}$$

(i) We write $T_2^2\tilde{T}_1^{-2} = \tilde{T}_2^2\tilde{W}^{-2}\tilde{W}^2\tilde{T}_1^{-2}$ for $W = (T_1)_{T_2^{-1}}$ which creates a triangle with $\tilde{T}_1$ and $\tilde{T}_2$ (see Fig. 5.1) We use Claim 5.5 twice – first when we take W, T_2, T_1 instead of X, Y, T from Claim 5.5(i), and second when we take T_1, W, T_2 instead of X, Y, T_2 from Claim 5.5(ii). By Claim 5.5(i) $\tilde{W}^2\tilde{T}_2^{-2} = \xi_1$, and thus $\tilde{T}_2^2\tilde{W}^{-2} = \xi_1^{-1}$. By Claim 5.5(ii) $\tilde{T}_1^{-2}\tilde{W}^2 = \xi_2$, and since by Claim 5.3(iii) $[\tilde{T}_1^{-2}, \tilde{W}^2] = c$, we get $\tilde{W}^2\tilde{T}_1^{-2} = c\xi_2$. Together we get (i).

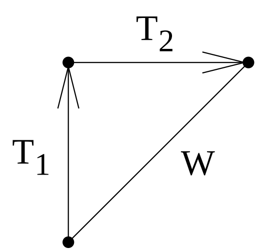

FIGURE 5.1

(ii) We write $\tilde{T}_3^2\tilde{T}_1^{-2} = (\tilde{T}_3^2\tilde{Z}^{-2})(\tilde{Z}^2\tilde{T}_1^{-2})$ for $Z = (\tilde{T}_1)_{\tilde{T}_3}$, which creates a triangle with T_1 and T_3. (See Fig. 5.2) By Claim 5.5 applied twice, $\tilde{T}_3^{-2}\tilde{Z}^2 = \xi_1$

and $\tilde{Z}^{-2}\tilde{T}_1^2 = \xi_3$. Thus $\tilde{Z}^{-2}\tilde{T}_3^2 = \xi_1^{-1}$ and $\tilde{T}_1^{-2}\tilde{Z}^2 = \xi_3^{-1}$. By 5.3(iii) we get $\tilde{T}_3^2\tilde{Z}^{-2} = c\xi_1^{-1}$, $\tilde{Z}^2\tilde{T}_1^{-2} = c\xi_3^{-1}$. Since $c \in \mathrm{Center}(G_9)$ and $c^2 = 1$, then $\tilde{T}_3^2\tilde{T}_1^{-2} = \xi_1^{-1}\xi_3^{-1}$.

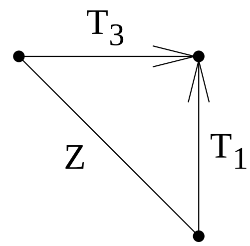

FIGURE 5.2

(iii) We install $\tilde{T}_3^{-2}\tilde{T}_3^2$ in the middle. Since $\tilde{T}_3$ relates to $\tilde{T}_5$ as $\tilde{T}_1$ relates to $\tilde{T}_3$ we have $\tilde{T}_5^2\tilde{T}_3^{-2} = \xi_3^{-1}\xi_5^{-1}$. Then $\tilde{T}_5^2\tilde{T}_3^{-2}\tilde{T}_3^2\tilde{T}_1^{-2} = \xi_3^{-1}\xi_5^{-1}\xi_1^{-1}\xi_3^{-1}$. Since $[\xi_3, \xi_5] = [\xi_3, \xi_1] = c$, $[\xi_1, \xi_5] = 1$, $c^2 = 1$, and $c \in \mathrm{Center}(G_9)$, we get $\tilde{T}_5^2\tilde{T}_1^{-2} = \xi_1^{-1}\xi_3^{-2}\xi_5^{-1}$.

(iv) We install $\tilde{T}_5^{-2}\tilde{T}_5^2$ in the middle. Since $\tilde{T}_9$ relates to $\tilde{T}_5$ as $\tilde{T}_3$ relates to $\tilde{T}_1$, then $\tilde{T}_9^2\tilde{T}_5^{-2} = \xi_5^{-1}\xi_9^{-1}$. Thus $\tilde{T}_9^2\tilde{T}_1^{-2} = \tilde{T}_9^2\tilde{T}_5^{-2}\tilde{T}_5^2\tilde{T}_1^{-2} = \xi_5^{-1}\xi_9^{-1}\xi_1^{-1}\xi_3^{-2}\xi_5^{-1} = c\xi_1^{-1}\xi_3^{-2}\xi_5^{-2}\xi_9^{-1}$. The last equation is based on the commutators of ξ_i and the fact that $c \in \mathrm{Center}\, G_9$, $c^2 = 1$.

(v) $\tilde{T}_8^2\tilde{T}_1^{-2} = \tilde{T}_8^2\tilde{T}_5^{-2}\tilde{T}_5^2\tilde{T}_1^{-2}$. T_8 relates to T_5 as T_2 relates to T_1 and thus $\tilde{T}_8^2\tilde{T}_5^{-2} = c\xi_5^{-1}\xi_8$. Thus $\tilde{T}_8^2\tilde{T}_1^{-2} = c\xi_5^{-1}\xi_8\xi_1^{-1}\xi_3^{-2}\xi_5^{-1}$. Since ξ_8 commutes with ξ_1, ξ_3 and ξ_5 commutes with ξ_1 and $[\xi_5, \xi_8] = [\xi_5, \xi_3] = c$, we have $\tilde{T}_8^2\tilde{T}_1^{-2} = c^4\xi_1^{-1}\xi_3^{-2}\xi_5^{-2}\xi_8$ which equals $\xi_1^{-1}\xi_3^{-2}\xi_5^{-2}\xi_8$.

(vi) $\tilde{T}_7^2\tilde{T}_1^{-2} = \tilde{T}_7^2\tilde{T}_8^{-2}\tilde{T}_8^2\tilde{T}_1^{-2}$. Since T_7 relates to T_8 as T_1 relates to T_3, then $\tilde{T}_8^2 T_7^{-2} = \xi_7^{-1}\xi_8^{-1}$ and $\tilde{T}_7^2\tilde{T}_8^{-2} = \left(\xi_7^{-1}\xi_8^{-1}\right)^{-1} = \xi_8\xi_7$ and thus $\tilde{T}_7^2\tilde{T}_1^{-2} = \tilde{T}_7^2\tilde{T}_8^{-2}\tilde{T}_8^2\tilde{T}_1^{-2} = (\xi_8\xi_7) \cdot \xi_1^{-1}\xi_3^{-2}\xi_5^{-2}\xi_8$. As before, $\tilde{T}_7^2\tilde{T}_1^{-2} = c^3\xi_1^{-1}\xi_3^{-2}\xi_5^{-2}\xi_7\xi_8^2$ which equals $c\xi_1^{-1}\xi_3^{-2}\xi_5^{-2}\xi_7\xi_8^2$.

(vii) $\tilde{T}_6^2\tilde{T}_1^2 = \tilde{T}_6^2\tilde{T}_7^{-2}\tilde{T}_7^2\tilde{T}_1^{-2}$. T_6 relates to T_7 as T_7 relates to T_8 and thus $\tilde{T}_6^2\tilde{T}_7^{-2} = \xi_7\xi_6$ (see (vi)). Therefore $\tilde{T}_6^2\tilde{T}_1^2 = c\xi_7\xi_6\xi_1^{-1}\xi_3^{-2}\xi_5^{-2}\xi_7\xi_8^2$ which equals as before $c^2\xi_1^{-1}\xi_3^{-2}\xi_5^{-2}\xi_6\xi_7^2\xi_8^2 = \xi_1^{-1}\xi_3^{-2}\xi_5^{-2}\xi_6\xi_7^2\xi_8^2$.

(viii) $T_4 = (T_5)_{T_8^{-1}T_7T_3^{-1}T_2}$. By Claim II.1.0 of [MoTe9], if $X = H(x)$ is represented by a diffeomorphism β and $Y = H(y)$, then $Y_X = H((y)\beta)$. Therefore

T_4 is a half-twist and we write $T_4 = H(t_4)$. Moreover, to describe T_4 we must apply $T_8^{-1}, T_7, T_3^{-1}, T_2$ on t_5 and we get t_4 is as in Figure 5.3:

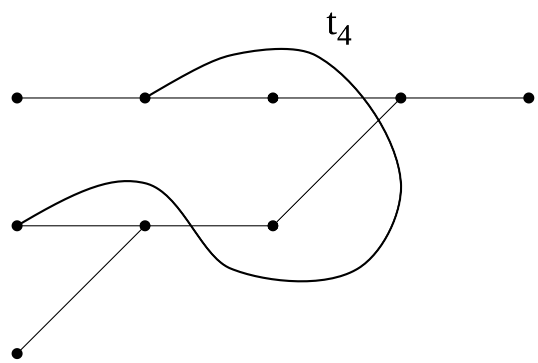

FIGURE 5.3

Since T_6 relates to T_4 as T_2 relates to T_1,then $\tilde{T}_6^2\tilde{T}_4^{-2} = c\xi_4^{-1}\xi_6$. Now $\tilde{T}_4^2\tilde{T}_1^{-2} = \tilde{T}_4^2\tilde{T}_6^{-2}\tilde{T}_6^2\tilde{T}_1^{-2} = (c\xi_4^{-1}\xi_6)^{-1}\xi_1^{-1}\xi_3^{-2}\xi_5^{-2}\xi_6\xi_7^2\xi_8^2$ which equals as before to $\xi_1^{-1}\xi_3^{-2}\xi_4\xi_5^{-2}\xi_7^2\xi_8^2$.

□ for Lemma 5.6

In order to express g_4 and ξ_4 in terms of $\{g_i\}_{i=1\ i\neq 4}^9$ and $\{\xi_i\}_{i=1\ i\neq 4}^9$, we need the following claim from [MoTe9]

5.7 Claim. *For $f_i = g_i$ or ξ_i*

$$(f_i)_{\tilde{T}_k} = \begin{cases} f_i^{-1}\nu & k = i \\ f_i & T_i, T_k \quad \textit{weakly disjoint} \\ f_k f_i & T_i, T_k \quad \textit{orderly adjacent} \\ f_i f_k^{-1} & T_i, T_k \quad \textit{are not orderly adjacent.} \end{cases}$$

$$(f_i)_{\tilde{T}_k^{-1}} = \begin{cases} f_i^{-1}\nu & k = i \\ f_i & T_i, T_k \quad \textit{weakly disjoint} \\ f_i f_k & T_i, T_k \quad \textit{orderly adjacent} \\ f_k^{-1} f_i & T_i, T_k \quad \textit{are not orderly adjacent.} \end{cases}$$

Proof. [MoTe9], Lemma IV.6.3. The conjugations for g_i are part of the definition of $G_0(9)$ (see 4.6) and remains when moving to $G_9 = \dfrac{\tilde{B}_9 \ltimes G_0(9)}{N_9}$.

□ for Claim 5.7

Now we can express g_4, ξ_4 in terms of $\{g_i\}_{i=1\ i\neq 4}^9$.

5.8 Claim.

(i) $\xi_4 = c\xi_2\xi_3\xi_5\xi_7^{-1}\xi_8^{-1}$

(ii) $g_4 = cg_2g_3g_5g_7^{-1}g_8^{-1}$

Proof. The proof is similar for (i) and (ii), and is based on 5.9. We shall only prove (i). By 5.2, $\xi_4 = (\xi_5)_{\tilde{T}_8^{-1}\tilde{T}_7\tilde{T}_3^{-1}\tilde{T}_2}$. Since T_5 and T_8 are not orderly adjacent by 5.7, $(\xi_5)_{\tilde{T}_8^{-1}} = \xi_8^{-1}\xi_5$. Since T_5 and T_7 are s disjoint, $(\xi_5)_{\tilde{T}_7} = \xi_5$. Since T_7 and T_8 are orderly adjacent, $(\xi_8)_{\tilde{T}_7} = \xi_7\xi_8$ and thus $(\xi_8^{-1})_{\tilde{T}_7} = \xi_8^{-1}\xi_7^{-1}$. Together we have $(\xi_5)_{\tilde{T}_8^{-1}\tilde{T}_7} = \xi_8^{-1}\xi_7^{-1}\xi_5$. We now apply $\tilde{T}_3^{-1}$. Since T_3 is disjoint from T_7 and T_8, then $(\xi_8^{-1}\xi_7^{-1})_{\tilde{T}_3^{-1}} = \xi_8^{-1}\xi_7^{-1}$. On the other hand, T_5 and T_3 are orderly adjacent and thus $(\xi_5)_{\tilde{T}_3^{-1}} = \xi_5\xi_3$ and $(\xi_5)_{\tilde{T}_8^{-1}\tilde{T}_7\tilde{T}_3^{-1}} = \xi_8^{-1}\xi_7^{-1}\xi_5\xi_3$. Now $\tilde{T}_2$ acts on ξ_3 to get $\xi_2\xi_3$ and does not move the other factors. Thus $\xi_4 = \xi_8^{-1}\xi_7^{-1}\xi_5\xi_2\xi_3$. We rearrange the factors using the comutators of 4.13 to get

$$\xi_4 = c\xi_2\xi_3\xi_5\xi_7^{-1}\xi_8^{-1}.$$

□ for Claim 5.8

In fact we are interested in δ up to a product with c and thus we formulate the following:

5.9 Claim. *Up to multiplication by c*

$$\delta = g_1g_2^{-2}g_5^{-2}g_6g_7^2g_9^{-1}\xi_1^{-8}\xi_2^4\xi_3^{-12}\xi_5^{-8}\xi_6\xi_7^2\xi_8^6\xi_9^{-1}\tilde{T}_1^{18}.$$

Proof of Claim 5.9. By definition, $\delta = \hat{\beta}\left(\prod_{i=9}^{1}\Gamma_{i'}\Gamma_i\right)$ which equals $\prod_{i=9}^{1}\hat{\beta}(\Gamma_{i'}\Gamma_i)$. We substitute in the product the values of $\hat{\beta}(\Gamma_{i'}\Gamma_i)$, $i = 1,\ldots,9$ from Claim 5.4. In the resulting formula, we replace $\tilde{T}_i^2$ by $\left(\tilde{T}_i^2\tilde{T}_1^{-2}\right)\tilde{T}_1^2$ for each $i = 1,\ldots,9$. We then substitute the formula for $\tilde{T}_i^2\tilde{T}_1^{-2}$ from Claim 5.6. We also substitute the values of g_4 and ξ_4 from Claim 5.8. We then get a formula for δ as a product of $\{\xi_i, g_i\}_{i=1\ i\neq 4}^{9}$ and $\tilde{T}_1^2$. Since we are not interested in the appropriate power of c, we can use Claim 5.3(ii) by "pushing" all the powers of $\tilde{T}_1^2$ to the right end of the last formula and rearrange the ξ_i's and the g_i'is to the get the claim.

□ for Claim 5.9

5.10 Claim. *Up to multiplication by c, for ζ_i and ξ_i from 4.13, we have* $\delta = \zeta_1\zeta_2\zeta_5\zeta_6(\zeta_7\zeta_9)^{-1}\xi_1^{-7}\xi_2^2\xi_3^{-12}\ xi_5^{-10}\xi_6^2\xi_7^4\xi_8^6\xi_9^{-2}\tilde{T}_1^{18}$.

Proof of Claim 5.10. In the formula from Claim 5.9 we replace g_i by $\zeta_i\xi_i$. Since $H_9' = \{1, c\}$ where $c \in \mathrm{Center}(G_9)$, we can rearrange the terms of the formula. Also using $\zeta_i^3 = 1$ we get the claim. $\square$ for Claim 5.10

We go back to the proof the theorem.

To prove that $\overline{G}_9$ is an extension of a solvable group by a symmetric group, it is enough to find a normal subgroup whose quotient is S_9. The subgroup will be $\overline{H}_9$.

Recall (3.12) that there exists a series $1 \lhd H_{9,0}' \lhd H_{9,0} \lhd H_9 \lhd G_9$. We defined $\overline{G}_9 = \dfrac{G_9}{\langle\delta\rangle}$, and $\overline{H}_9$, $\overline{H}_{9,0}$, $\overline{H}_{9,0}'$ to be the images of H_9, $H_{9,0}$, $H_{9,0}'$ in $\overline{G}_9$ respectively, and we have a series $1 \lhd \overline{H}_{9,0}' \lhd \overline{H}_{9,0} \lhd \overline{H}_9 \lhd \overline{G}_9$. We shall compute the quotients. Since $\delta \in H_9$ (Claim 5.1), $\dfrac{\overline{G}_9}{\overline{H}_9} \simeq \dfrac{G_9}{H_9} \simeq S_9$. Since $\dfrac{H_9}{H_{9,0}}$ is generated by $\tilde{T}_1^2$ (see 4.14), $\dfrac{\overline{H}_9}{\overline{H}_{9,0}}$ is also generated by $\tilde{T}_1^2$. By Claim 5.12, $(\tilde{T}_1^2)^9\delta^{-1} \in H_{9,0}$. So when considered as elements of $\overline{H}_9$, $(\tilde{T}_1^2)^9 \in \overline{H}_{9,0}$, and thus as elements of $\dfrac{\overline{H}_9}{\overline{H}_{9,0}}$, $\tilde{T}_1^2$ is of order 9. Thus $\dfrac{\overline{H}_9}{\overline{H}_{9,0}} \simeq \mathbb{Z}_9$. Now let $Y_1 = \xi_1^{-7}\xi_2^2\xi_3^{-12}\xi_5^{-10}\xi_6^2\xi_7^4\xi_8^6\xi_9^{-2}(\tilde{T}_1)^{18}$. We complete Y_1 to a base $Y_1, \ldots, Y_9$ of $\langle \tilde{T}_1^2, \{\xi_i\}_{i=1\ i\neq 4}^9\rangle$. $\dfrac{H_9}{H_9'}$ is generated by $Y_1, \ldots, Y_9, \{\zeta_i\}_{i=1\ i\neq 4}^2$. Modulo $\langle\delta\rangle$, $Y_1 \in \{\zeta_i\}_{i=1\ i\neq 4}^9$ and $Y_2, \ldots, Y_8$ are of infinite order. Thus $\dfrac{\overline{H}_{9,0}}{\overline{H}_{9,0}'} \simeq (\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z})^8$.

By 3.2, $\langle\delta\rangle$ is an infinite cyclic group where $\delta \in \mathrm{Center}(G_9)$. Since $c^2 = 1$, then $\langle c\rangle \cap \langle\delta\rangle = 1$ and $\overline{H}_{9,0}' \simeq \dfrac{H_{9,0}'}{\langle\delta\rangle \cap H_{9,0}'} \sim H_{9,0}' \simeq \mathbb{Z}_2$. Thus we have a series $1 \lhd \overline{H}_{9,0}' \lhd \overline{H}_{9,0} \lhd \overline{H}_9 \lhd \overline{G}_9$ s.t. $\overline{H}_9$ is a solvable group, and $\dfrac{\overline{G}_9}{\overline{H}_9} \simeq$ symmetric group of 9 elements . $\square$ for Theorem 5.0

§6. The result of Veronese of order p.

The result for any Veronese is similar (see below), but the representation of the proof for $p = 3$ is much more "reader friendly". The result for any p is as follows:

There exist 2 series $1 \triangleleft A \triangleleft B \triangleleft C \triangleleft G$ and $1 \triangleleft \overline{A} \triangleleft \overline{B} \triangleleft \overline{C} \triangleleft \overline{G}$ s.t.

$$G/C \simeq \overline{G}/\overline{C} \simeq S_{p^2}$$

$$C/B \simeq \mathbb{Z},\ \overline{C}/\overline{B} \simeq Z_q$$

$$B/A \simeq \overline{B}/\overline{A} \simeq \begin{cases} (\mathbb{Z} \oplus \mathbb{Z}_3)^{p^2-1} & p = O(3) \\ \mathbb{Z}^{p^2-1} & p \not\equiv O(3) \end{cases}$$

$$A \simeq \overline{A} \simeq \begin{cases} \mathbb{Z}_2 & p \text{ odd} \\ 0 & p \text{ even} \end{cases}$$

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